Kinetic Equations

Text of the Exercises

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Exercise 1

Let $f : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a function which is $\mathcal{C}^1$ in t and x, such that $(\omega, v_*) \mapsto B(v - v_*, \omega)\big(f' f'_* - f f_*\big)$ is integrable for all (t, x, v).

(i) Assume that f solves the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{1}$$

with initial datum f_0. Prove that

$$f(t,x,v) = f_0(x - tv, v) + \int_0^t Q(f,f)(s, x - (t-s)v, v) ds. \tag{2}$$

In the following we will call a continuous function f which is solution of (2) a *mild solution* of the Boltzmann equation.

We consider now a system of *Maxwellian molecules*, i.e. a system in which B is of the form $B(v - v_*, \omega) = b(\cos\theta)$, where we indicate with θ the angle between ω and the vector $v - v_*$. On b we only assume that $\int_{\mathbb{S}^2} b(\cos\theta)\, d\omega$ is finite and bounded uniformly in v and v_* (notice that by definition θ depends on ω and v_*), i.e. there exists a positive real number β such that

$$\int_{\mathbb{S}^2} b(\cos\theta)\, d\omega \leqslant \beta, \qquad \forall\, (v, v_*) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{3}$$

We also call $\varphi(v) := e^{-\alpha |v|^2}$, with $\alpha > 0$, a Maxwellian function and $M := \int_{\mathbb{R}^3} \varphi(v)\, dv$ its mass.

Finally, assume that $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that $|f_0| \leqslant \varphi$ and define the sequence of functions $\left\{\widetilde{f}_n\right\}_{n \geqslant 0}$ defined recursively as

$$\begin{cases} \widetilde{f}_0(t,x,v) = f_0(x,v), \\ \widetilde{f}_{n+1}(t,x,v) = f_0(x - tv, v) + \int_0^t Q(\widetilde{f}_n, \widetilde{f}_n)(s, x - (t-s)v, v) ds. \end{cases} \tag{4}$$

(ii) Assuming that $|\widetilde{f}_n(t,x,v)| \leqslant 2\varphi(v)$, prove that $|\widetilde{f}_{n+1}(t,x,v)| \leqslant (1 + 8\beta M t)\varphi(v)$, for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, and all $n \geqslant 0$. Prove also that $\widetilde{f}_n$ is continuous, for all $n \geqslant 0$.

(iii) We define $T = 1/(8\beta M)$. Prove that, for all $t \in [0,T]$ and $n \geqslant 0$,

$$|\widetilde{f}_n(t, \cdot, \cdot)| \leqslant 2\varphi. \tag{5}$$

(iv) Define

$$\|f\|_{\varphi} := \sup_{(t,x,v)\in[0,T]\times\mathbb{R}^3\times\mathbb{R}^3} \frac{|f(t,x,v)|}{\varphi(v)}, \tag{6}$$

where it's important to notice that now the supremum in t is taken on the interval $[0,T]$.

Denote $[f](v,v_*,\omega) = f(v')f(v'_*) - f(v)f(v_*)$ (where we omit everywhere the variable x out of convenience) and prove that, for all $x, v, v_* \in \mathbb{R}^3$, $t \in [0,T]$, $\omega \in \mathbb{S}^2$ and all $n \geqslant 0$:

$$\frac{\left|[\widetilde{f}_{n+1}](v,v_*) - [\widetilde{f}_n](v,v_*)\right|}{\varphi(v)\varphi(v_*)} \leqslant 8\|\widetilde{f}_{n+1} - \widetilde{f}_n\|_{\varphi}. \tag{7}$$

(v) Use (7) to bound $|Q(\widetilde{f}_{n+1},\widetilde{f}_{n+1})(v) - Q(\widetilde{f}_n,\widetilde{f}_n)(v)|$ and to deduce that for all $t \in [0,T]$

$$\|Q(\widetilde{f}_{n+1},\widetilde{f}_{n+1}) - Q(\widetilde{f}_n,\widetilde{f}_n)\|_{\varphi} \leqslant 8\beta M\|\widetilde{f}_{n+1} - \widetilde{f}_n\|_{\varphi}. \tag{8}$$

(vi) Prove, for all $n \geqslant 1$, that

$$\begin{cases} \|\widetilde{f}_{n+1} - \widetilde{f}_n\|_{\varphi} \leqslant 8\beta MT\|\widetilde{f}_n - \widetilde{f}_{n-1}\|_{\varphi}, \\ \|Q(\widetilde{f}_{n+1},\widetilde{f}_{n+1}) - Q(\widetilde{f}_n,\widetilde{f}_n)\|_{\varphi} \leqslant 8\beta MT\|Q(\widetilde{f}_n,\widetilde{f}_n) - Q(\widetilde{f}_{n-1},\widetilde{f}_{n-1})\|_{\varphi}. \end{cases} \tag{9}$$

Deduce, for any $0 < \alpha < 1$, that the sequences of functions $\{\widetilde{f}_n\}_{n\geqslant 0}$ and $\left\{Q(\widetilde{f}_n,\widetilde{f}_n)\right\}_{n\geqslant 0}$ are respectively converging uniformly towards some continuous limits f and $\widetilde{Q}$ on $[0,\alpha T] \times \mathbb{R}^3 \times \mathbb{R}^3$.

(vii) Prove that f is a mild solution of the Boltzmann equation with initial datum f_0.

Remark. *We recall that, for particles interacting via inverse-power laws potentials $\phi(r) = 1/r^{k-1}$ (with $k > 2$), the collision kernel $B(v - v_*, \cos\theta)$ takes the particular form $B = b(\cos\theta)|v - v_*|^{\gamma}$, with $\gamma = (k-5)/(k-1)$, and b locally smooth. The case we just considered is the case of Maxwellian molecules, corresponding to the case $\gamma = 0$.*

Exercise 2

Note that to make sense, a mild solution of the Boltzmann equation, as defined by (2), does not need to be differentiable, with respect to any of its variables.
Using the result of the previous exercise, providing a (local in time) mild solution f to the Boltzmann equation such that $|f| \leqslant 2\varphi$, prove that

$$\lim_{h\to 0} \frac{1}{h}\big(f(t+h, x+hv, v) - f(t,x,v)\big) \tag{10}$$

makes sense for all fixed $(t,x,v) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$.

For a general function f which is $\mathcal{C}^1$ in t and x , what is the limit, when h goes to zero, of the quantity (10)?

Exercise 3

In the case of hard spheres, the loss term of the Boltzmann equation writes $L(f)(v)f(v)$, where

$$L(f)(v) = \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |\omega \cdot (v - v_*)| f(v_*) dv_* \, d\omega. \tag{11}$$

Denote now as φ_α the Maxwellian function $e^{-\alpha|v|^2}$.

In their famous article of 1978, Kaniel and Shinbrot introduced the following cathegorization on L: if there exists a positive constant $C(\alpha)$ depending only on α and a positive number $0 \leqslant \lambda < 2$ such that, for all $v \in \mathbb{R}^3$

$$L(\varphi_\alpha)(v) \leqslant C(\alpha)(1 + |v|^\lambda), \tag{12}$$

the collision kernel B describes a *soft interaction* if $\lambda = 0$, and it describes a *hard interaction* if $\lambda > 0$.

(i) Show that in the case of the hard spheres, condition (12) holds for $\lambda = 1$, that is one can find a constant $C(\alpha)$ such that (12) holds for all $v \in \mathbb{R}^3$.

(ii) One may wonder if this control can be improved in the case of the hard spheres. Show that we cannot choose $\lambda = 0$ (so that, of course, the hard sphere collision kernel does *not* represent a soft interaction).

(iii) Show that (12) does not hold for any $0 < \lambda < 1$ in the hard sphere case.